UDC 539. 3

BENDING OF THICK PLATES UNDER AN ARBITRARY LOAD

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A problem of bending of a plate under an arbitrary load is studied. A method proposed by Lur'e [1] is employed, which reduces the three-dimensional problems of the plate theory to the two-dimensional problems using the infinite order differential operators of the displacements and rotations of the middle plane of the plate. Three stress functions are introduced, and these make it possible to consider the effects of the normal and tangential loads separately. Approximate equations are constructed using the method of homogeneous solutions [2]. A problem of axisymmetric bending of a plate under the action of tangential forces is solved using the first approximation equations.

1. Introduction of the stress functions. The equations of equilibrium of a thick plate bent by an arbitrary load distributed over its ends, have the form [1]

$$\left[\cos hD - \frac{m}{2(m-1)}\frac{h\sin hD}{D}\partial_1^2\right]u_0' + \left[\frac{m}{2(m-1)}\frac{h\sin hD}{D}\partial_1\partial_2\right]v_0' + \partial_1\left[\cos hD + \frac{m}{2(m-1)}hD\sin hD\right]w_0 = \frac{t_x}{2\mu} \tag{1.1}$$

$$\left[-\frac{m}{2(m-1)}\frac{h\sin hD}{D}\partial_1\partial_2\right]u_0' + \left[\cos hD - \frac{m}{2(m-1)}\frac{h\sin hD}{D}\partial_2^2\right]v_0' + \partial_2\left[\cos hD + \frac{m}{2(m-1)}hD\sin hD\right]w_0 = \frac{t_y}{2\mu}$$

$$-\partial_1\left[\frac{m-2}{2(m-1)}\frac{\sin hD}{D} + \frac{m}{2(m-1)}h\cos hD\right]u_0' - \partial_2\left[\frac{m-2}{2(m-1)}\frac{\sin hD}{D} + \frac{m}{2(m-1)}h\cos hD\right]v_0' + \left[-\frac{3m-2}{2(m-1)}D\sin hD + \frac{m}{2(m-1)}hD^2\cos hD\right]w_0 = \frac{p}{2\mu}$$

$$\Delta = D^2 = \partial_1^2 + \partial_2^2, \quad \partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad u_0' = \left.\frac{\partial u}{\partial z}\right|_{z=0}$$

$$v_0' = \left.\frac{\partial v}{\partial z}\right|_{z=0}$$

$$(\tau_{xz} = {}^1/_2\, t_x,\ \tau_{yz} = {}^1/_2\, t_y,\ \sigma_z = \pm {}^1/_2\, p \text{ при } z = \pm h)$$

Here m is the Poisson's ratio, μ is the shear modulus, w_0 is the deflection while u_0' and v_0' denote the " rotations" of the middle plane of the plate.

Let us perform a change of variables intended to separate the potential and vortical components of the tangential load t_x and t_y and the rotations u_0' and v_0'

$$t_x = -\partial_1 q_1 + \partial_2 q_2, \quad t_y = -\partial_2 q_1 - \partial_1 q_2 \tag{1.2}$$

$$u_0' = -\partial_1 \varphi^* + \partial_2 \Psi^*, \quad v_0' = -\partial_2 \varphi^* - \partial_1 \Psi^* \tag{1.3}$$

The first two equations of (1.1) can be written in the form

$$-\partial_1 \gamma_1 + \partial_2 \gamma_2 = 0, \quad \partial_2 \gamma_1 + \partial_1 \gamma_2 = 0$$

$$\gamma_1 = M^- \varphi^* - M^+ w_0 - \frac{1}{2\mu} q_1, \quad \gamma_2 = \cos hD\Psi^* - \frac{1}{2} q_2$$

$$M^\pm = \cos hD \pm \frac{m}{2(m-1)} hD \sin hD$$

where γ_1 and γ_2 denote conjugate harmonic functions.

Setting now $\varphi^* = \varphi + \gamma_1$, $\Psi^* = \Psi + \gamma_2$ and assuming that for an arbitrary harmonic function γ

$$\cos hD\gamma = \gamma, \quad hD \sin hD\gamma = 0$$

we reduce the system (1.1) to the form

$$M^- \varphi - M^+ w_0 = \frac{q_1}{2\mu}, \quad M_1 \varphi + M_2 w_0 = \frac{p}{2\mu} \tag{1.4}$$

$$M_1 = \frac{1}{2(m-1)} [(m-2) D \sin hD + mhD^2 \cos hD],$$

$$M_2 = \frac{1}{2(m-1)} [-(3m-2) D \sin hD + mhD^2 \cos hD]$$

$$\cos hD \, \Psi = \frac{2}{2\mu} \tag{1.5}$$

Introducing now the operator determinant of the equation (1.4), we arrive at the following equations for the stress functions Φ_1 and Φ_2:

$$\frac{mh}{m-1} D^2 \left(1 - \frac{\sin 2hD}{2hD}\right) \Phi_1 = \frac{q_1}{2\mu} \tag{1.6}$$

$$\frac{mh}{m-1} D^2 \left(1 - \frac{\sin 2hD}{2hD}\right) \Phi_2 = \frac{p}{2\mu} \tag{1.7}$$

The unknown functions φ, u_0', v_0' and w_0 are determined from the formulas

$$\varphi = M_2 \Phi_1 + M^+ \Phi_2, \quad w_0 = M_1 \Phi_1 + M^- \Phi_2$$

$$u_0' = -\partial_1 (M_2 \Phi_1 + M^+ \Phi_2) + \partial_2 \Psi,$$

$$v_0' = -\partial_2 (M_2 \Phi_1 + M^+ \Phi_2) - \partial_1 \Psi$$

In this manner we have reduced the system (1.1) to three separate equations (1.5)-(1.7) for the functions Φ_1, Φ_2 and Ψ. Setting $t_x = t_y = 0$ we obtain $\Phi_1 = 0$ and the Eqs. (1.5) and (1.10) together with the expressions for φ, u_0', v_0' and w_0 will fully coincide with the equations obtained earlier [2] for the bending of a plate acted upon by a normal load only.

Below we give the geometrical and the force boundary conditions for a thick plate. The sense of the geometrical conditions is obvious. Since the displacements of the points of the plate are taken in the form of solutions of the equations of the theory of elasticiity for a layer and have the form

$$u = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \left[\Delta^n u_0' + \frac{nm}{2(m-1)} \partial_1 \Delta^{n-1} \vartheta_0' \right] \tag{1.9}$$

$$v = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \left[\Delta^n v_0' + \frac{nm}{2(m-1)} \partial_2 \Delta^{n-1} \vartheta_0' \right]$$

$$w = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \left[\Delta^n w_0 + \frac{nm}{2(m-1)} \Delta^{n-1} \vartheta_0' \right]$$

$$\vartheta_0' = \partial_1 u_0' + \partial_2 v_0' - \Delta w_0$$

it follows that the displacements u, v and w will vanish at the contour when the coefficients of their expansions in powers of z are equal to zero. For a rigidly clamped edge these conditions are

$$u_0' = 0, \quad v_0' = 0, \quad w_0 = 0, \quad \Delta^n u_0' + \frac{nm}{2(m-1)} \partial_1 \Delta^{n-1} \vartheta_0' = 0 \tag{1.10}$$

$$\Delta^n v_0' + \frac{nm}{2(m-1)} \partial_2 \Delta^{n-1} \vartheta_0' = 0, \quad \Delta^n w_0 + \frac{nm}{2(m-1)} \Delta^{n-1} \vartheta_0' = 0$$
$$(n = 1, 2, \ldots)$$

The force boundary conditions are obtained from the principle of the minimum of the potential energy, and for the free edge they have the form

$$G_1^{(n)} n_x + H^{(n)} n_y = 0, \quad G_2^{(n)} n_y + H^{(n)} n_x = 0 \tag{1.11}$$

$$N_1^{(n)} n_x + N_2^{(n)} n_y = 0 \quad (n = 0, 1, 2 \ldots)$$

Here $G_1^{(0)}$, $G_2^{(0)}$, $H^{(0)}$, $N_1^{(0)}$, $N_2^{(0)}$ are the bending and torsional moments and transverse forces, and $G_1^{(n)}$, $G_2^{(n)}$, $H^{(n)}$, $N_1^{(n)}$, $N_2^{(n)}$ $(n = 1,2,\ldots)$ are their hyperstatic analogs, i. e. multimoments. The stress characteristics mentioned above are found from the following formulas [3]:

$$G_1^{(n)} = \frac{(-1)^n}{(2n+1)!} \int_{-h}^{h} \sigma_x z^{2n+1} dz, \quad N_1^{(n)} = \frac{(-1)^n}{(2n)!} \int_{-h}^{h} \tau_{zx} z^{2n} dz \tag{1.12}$$

Replacing in (1.12) σ_x by σ_y or τ_{xy} and τ_{zx} by τ_{zy} we obtain the corresponding expressions for $G_2^{(n)}$ or $H^{(n)}$ and $N_2^{(n)}$. The boundary conditions (1.10) and (1.11) written in terms of the stress functions Φ_1, Φ_2 and Ψ using the formula (1.8).

2. **Use of the method homogeneous solutions to construct the approximate equations.** We use the method of homogeneous solutions to construct the approximate solutions. The differential operators in Eqs. (1.5) - (1.7) are written in the form of infinite products containing the roots of the following transcendental equations

$$\sin 2\rho = 2\rho, \quad \cos \rho = 0 \tag{2.1}$$

The first equation has a zero root and complex roots which can be arranged in sets of four roots with equal moduli, while the roots of the second equation are real and they all have identical moduli. Taking into account the properties of the roots of the transcendental equations (2.1), we write the left-hand sides of (1.5) - (1.7) in the form of infinite products, and we obtain

$$\prod_{n=1}^{\infty}\left[1-\frac{h^2\Delta}{[{}^1/_2(2n-1)\pi]^2}\right]\Psi=\frac{q_2}{2\mu} \tag{2.2}$$

$$D^\circ\Delta^2\prod_{n=1}^{\infty}H_n(\Delta)\Phi_1=q_1 \tag{2.3}$$

$$D^\circ\Delta^2\prod_{n=1}^{\infty}H_n(\Delta)\Phi_2=p \tag{2.4}$$

$$H_n(\Delta)=\left[1-\frac{h^2\Delta}{(\alpha_n+i\beta_n)^2}\right]\left[1-\frac{h^2\Delta}{(\alpha_n-i\beta_n)^2}\right],\quad D^\circ=\frac{4\mu h^3 m}{3(m-1)}$$

(D° denotes the cylindrical rigidity of the plate).

It should be remembered that we must solve two separate problems, the problem of bending of the plate under the action of tangential forces described by the equations (2.2) and (2.3), and the problem of bending of the plate under the action of a normal load described, as shown in [2], by the equation (2.4) and the homogeneous equation (2.2). By restricting the number of roots of (2.1) to a definite value, we obtain various systems of the approximate equations. Thus, considering the problem of bending by the tangential forces in its first approximation, we retain in (2.3) the terms corresponding to the null root in the first quarter of the complex conjugate roots and restrict ourselves in (2.2) to the roots $\rho_1=\pi/2$ and $\rho_2=3\pi/2$.

This yields

$$D^\circ\Delta^2 H_1(\Delta)\Phi_1=q_1,\quad \left(1-\frac{4h^2\Delta}{\pi^2}\right)\left(1-\frac{4h^2\Delta}{9\pi^2}\right)\Psi=q_2$$

The total order of the system Δ^6, and this makes it possible for six boundary conditions to be fulfilled at a time. In the case of a clamped edge these conditions are given by (1.10) with $n=0$ and $n=1$. The force boundary conditions will hold for the static, as well as for the hyperstatic first order characteristics. For a free edge the conditions are given by the relationtions (1.11) with $n=0$ and $n=1$.

The method of constructing further approximations of the order Δ^9, Δ^{12}, etc. is self-evident.

3. Axsymmetric bending of a circular plate under the action of tangential forces. We illustrate the theory expounded above by considering the problem of a circular plate radius $r=a$, with an axisymmetric system of uniformly distributed radial forces of intensity ${}^1/_2 t_r$, applied to both ends of the plate and thus causing the bending. At the top end the radial forces are applied from the edge towards the center, and in the opposite direction at the bottom end. The lateral surface of the plate is rigidly clamped.

From (1.8), (1.5) and (1.6) it follows that in the presence of axial symmetry the functions Φ_1 and Ψ can be found independently of each other using the polar (r, θ) coordinates. The unknown quantities appearing in the problem in question are completely established in terms of the function Φ_1. When $t_r=$ const.and $t_\theta=0$, the formulas (1.2) yield the value of the function of tangential load

$$q_1=t_r r+A_0,\quad A_0=\text{const}$$

In the first approximation the problem is described by the equation

$$D^\circ \Delta^2 H_1(\Delta)\,\Phi_1 = rt_r + A_0$$

Its general solution can be written in the form of a sum of a particular solution of a biharmonic function and of two complex conjugate functions satisfying the modified Bessel equations. Thus we obtain

$$\Phi_1 = \frac{t_r r^5}{225D^\circ} + \frac{A_0 r^4}{64D^\circ} + C_3 r^2 + (A_1 + iB_1)\, I_0\left(\frac{\alpha_1 + i\beta_1}{h} r\right) + (A_1 - iB_1)\, I_0\left(\frac{\alpha_1 - i\beta_1}{h} r\right)$$

(the additive constant is neglected, since all the quantities sought are expressed in terms of the derivatives of Φ_1 only).

The constants A_0, C_3, A_1 and B_1 are determined from the conditions of rigid clamping of the lateral surface of the plate. The conditions are expressed in the terms of the function Φ_1 by means of the formulas (1.9) and (1.11), and are

$$\frac{d}{dr}(M_2\Phi_1) = 0 \quad (u_{r0}' = 0), \quad M_1\Phi_1 = 0 \quad (w_0 = 0)$$

$$\frac{d}{dr}(M_3\Phi_1) = 0 \quad \left(u_{r0}' + \frac{m}{2(m-1)}\frac{d}{dr}\vartheta_0' = 0\right)$$

$$M_4\Phi_1 = 0 \quad \left(\Delta w_0 + \frac{m}{2(m-1)}\vartheta_0' = 0\right)$$

$$M_3 = \frac{1}{2(m-1)}[(5m-2)D^3 \sin hD - mhD^4 \cos hD]$$

$$M_4 = \frac{1}{2(m-1)}[(m+2)D^3 \sin hD - mhD^4 \cos hD]$$

for $r = a$

The numerical computations were carried out for the following data: $m = 3, \alpha_1 + i\beta_1 = 3.749 + i1.384$, $a / h = 5$. The asymptotic formulas for the Bessel functions can be used already when $a / h = 3$ and this was done. In all computations performed the coefficients A_1 and B_1 were not encountered; instead we dealt with their values multiplied by the quantity $\varkappa$ defined as follows:

$$\varkappa = \left(2\pi \frac{a}{h}\sqrt{\alpha_1^2 + \beta_1^2}\right)^{-1/2} \exp\frac{\alpha_1 a}{h} = 1.23\cdot 10^7$$

The following values were obtained for the unknown constants:

$$A_0 = -3.33 t_r h, \qquad A_1\varkappa = 3.16\cdot 10^{-4}\frac{t_r h^5}{D^\circ}$$

$$C_3 = 1.90\frac{t_r h^3}{D^\circ}, \qquad B_1\varkappa = 1.75\cdot 10^{-3}\frac{t_r h^5}{D^\circ}$$

and the above results were used to determine the deflection at the center of the plate

$$w_0^{(1)} = -8.99 t_r h^4 / D^\circ$$

Solving the same problem in the zero approximation, when the problem is described by a biharmonic equation for the function Φ_1 and only two conditions ($u_{r_0}' = 0$ and $w_0 = 0$) hold on the lateral surface, gives the following result:

$$w_0^{(0)} = -\frac{t_r h}{36D^\circ}\left[2a^3 + \frac{3(4m-1)}{m-1}ah^2\right] \tag{3.1}$$

and for $a / h = 5$ we have

$$w_0^{(0)} = -9.24 t_\gamma h^4 / D^\circ \qquad (3.2)$$

Comparing the results of the zero and first approximations we find, that the difference in the deflection is not large, but, that a more rigid clamping corresponding to the first approximation reduces the deflection at the center by about 3%. Computing the deflection of an analogously loaded plate according to the Kirchoff theory, we obtain

$$w_0 = -6{,}94 t_\gamma h^4 / D^\circ \qquad (3.3)$$

The value of w_0 defined by (3.3) can also be obtained from (3.1) by neglecting the term containing h^2.

The difference in the values of the deflection given by the formulas (3.2) and (3.3) can be explained as follows. In the Kirchoff theory of plates (the elementary theory) the effect of the tangential stresses on the bending [4] is neglected. In the zeroth and subsequent approximations of the proposed (multimoment) theory of bending plates, the deflection is determined with the tangential stresses taken into account and this leads to larger values of the deflection compared with the results obtained by means of the elementary theory of plates. Formally, this leads to different conditions at the lateral surface for a clamped plate in the elementary, and in the multimoment theory. The Kirchoff theory of plates which disregards the deformation due to transverse shear, demands the absence of rotation of the tangent plane towards the middle plane of the plate at the point at which the contour is clamped. The condition that u_{r0}' is equal to zero, which holds in the zeroth approximation of the multimoment theory, means that at the clamped edge the tangent to the middle of the linear element lying on the cylindrical boundary surface must remain perpendicular to the initial position of the middle plane. In the Love theory of thick plates [5] it was precisely such a condition that was used in analogy with the problem of a clamped beam. The axisymmetric problem of bending of a plate under a normal load was studied in [6].

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